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Formulation of non-standard dissipative behavior of geomaterials

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Abstract. In this paper, fundamental mathematical concepts for modeling the dissipative behavior of geomaterials are recalled. These concepts are illustrated on two basic models and applied to derive a new form of the evolution law of the modified Cam-clay model. The aim is to discuss the mathematical structure of the constitutive relationships and its consequences on the structural level. It is recalled that non-differentiable potentials provide an appropriate means of modeling rate-independent behavior. The Cam-clay model is revisited and a standard version is presented. It is seen that this standard version is non-dissipative, which at the same time explains why a non-standard version is needed. The partial normality is exploited and an implicit variational formulation of the modified Cam-clay model is derived. As a result, the solution of boundary-value problems can be replaced by seeking stationary points of a functional.

Key words: convex analysis, Fenchel transform, internal variables, modified Cam-clay model, variational inequality

1. Introduction

Developing a model for the inelastic behavior of geomaterials is usually carried out in a rather empirical fashion where experimental data are curve-fitted to derive the constitutive relations. The relevance of the model is then assessed on the basis of its capability to reproduce several key characteristics of material behavior. Nowadays, modeling has attained a high degree of sophistication allowing a fine description of the behavior of materials. As a result complex constitutive relations have emerged that have to be implemented in finite-element codes. Lately much attention has been paid to thermodynamic consistency of constitutive models (see [1] for an application to geotechnical materials). The two principles of thermodynamics are used to validate or reject models if they fail to pass the thermodynamic test. This aspect is essential to ensure that constitutive models are physically consistent. However, the mathematical structure of the constitutive relations is crucial when questions such as existence and uniqueness of the boundary-value problems (BVP) need to be answered. Furthermore, convergence of numerical algorithms is closely related to properties of the constitutive operator. Unfortunately this aspect of the problem is rarely addressed, particularly in geomechanics.

Often constitutive relations are provided in a rather explicit form where the stress-like variables are given as a function of the strain-like variables. For instance, the viscoelastic strain rates are related to the stresses through a linear operator. This defines a mapping between the stress space and the strain rate space called constitutive operator. If the constitutive operator is invertible then the inverse relationship exists. Another example is the flow rule in associated plasticity where the plastic strain rates are related to the stresses through the so-called normality rule and the complementarity conditions. This formulation of the flow rule does not provide an explicit expression of the constitutive operator. The main difficulty stems from the non-smooth (multi-valued) character of this constitutive model. Using Convex Analysis tools, Moreau [2, 3] has shown that, under some conditions, the constitutive operator can be derived from a scalar-valued function that acts like a potential for the flow rule. The work of Moreau [2, 3] on the mathematical structure of mechanical laws is an important step in material modeling. The main contribution is probably a unified framework proposed for mechanical models including the multi-valued ones. The variational structure revealed by the "potential form" of the constitutive relationship prove to be useful regarding to the numerical and mathematical aspects of boundary-value problems. This property ensures the existence of stationary principles that becomes minimum principles if the functional is convex. Another key-step has been accomplished by Nguyen Quoc Son [4, 5] who extended Moreau's work to more complex multivalued laws ("visco"-plasticity with hardening, damage,$\cdots$) using the phenomenological approach with internal variables. However, geomaterials exhibit frictional behavior, they undergo plastic changes of volume and need to be modeled by plasticity theories with 'non-associated' (or non-standard) flow rule. It is shown that a non-standard model for the modified Cam-clay is not avoidable, otherwise it will lead to a non-dissipative model which contradicts experimental investigations (see [6]). A convenient formulation of this model based on implicit normality is discussed.

2. The constitutive operator

The phenomenological approach with internal variables provides a unified framework for developing various models arising in engineering applications. It consists of supplementing the deformation $\boldsymbol{\varepsilon}$ by a set of internal (strain-like) variables $\boldsymbol{\kappa} = (\boldsymbol{\kappa}_i,\ i = 1, \dots, n)$ which account for the internal restructuring taking place during the dissipative process. The number and the mathematical nature (tensor, vector or scalar) of the internal variables depend on the model under consideration. The notation used here will be one in which symmetric second-order tensors are represented as six-dimensional vectors and denoted by bold letters. More complex operator are capital doubled (*e.g.* $\mathbb{D}$ for Hooke's tensor). For the sake of a compact representation, internal variables $\boldsymbol{\kappa}_i$ are grouped together in a unique vector $\boldsymbol{\kappa} \in \mathbb{R}^m$ made by the following ordered n-tuples:

$$\boldsymbol{\kappa}^{\mathrm{t}} = \left[\boldsymbol{\kappa}_1^{\mathrm{t}}, \dots, \boldsymbol{\kappa}_i^{\mathrm{t}}, \dots, \boldsymbol{\kappa}_n^{\mathrm{t}}\right]$$

where "t" stands for the usual transposition, $\mathbb{R}^m$ is a m-dimensional vector space and $\boldsymbol{\kappa}_i^{\mathrm{t}}$ can be either a vectorial representation of a tensor, a vector or a scalar. The rate of an internal variable, also called velocity, is denoted by a superimposed dot. A set of generalized stresses $\boldsymbol{\pi} = (\boldsymbol{\pi}_i,\ i = 1, \dots, n)$, responsible for the internal modifications, are defined such that (generalized stresses) $\times$ (rate of change of internal variables) gives the rate of dissipation. Grouping together the generalized stresses in the vector $\boldsymbol{\pi} \in \mathbb{R}^m$, one may give the rate of dissipation as a scalar product in $\mathbb{R}^m$

$$\boldsymbol{\pi} \cdot \dot{\boldsymbol{\kappa}} = \boldsymbol{\pi}_1^{\mathrm{t}} \cdot \dot{\boldsymbol{\kappa}}_1 + \cdots + \boldsymbol{\pi}_i \cdot \dot{\boldsymbol{\kappa}}_i + \cdots + \boldsymbol{\pi}_n \cdot \dot{\boldsymbol{\kappa}}_n,$$

where a dot "$\circ \cdot \circ$" represents the usual scalar product. The m-dimensional linear space $\mathbb{R}^m$, whose elements are the velocities, is called the velocity space and denoted by $\mathcal{V}$. The bilinear form generated by the rate of dissipation puts the velocity space $\mathcal{V}$ in duality with the force space $\mathcal{F}$ comprising the generalized stresses $\boldsymbol{\pi}$:

$$\forall (\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi}) \in \mathcal{V} \times \mathcal{F} \mapsto \dot{\boldsymbol{\kappa}} \cdot \boldsymbol{\pi} \in \mathbb{R}. \tag{1}$$

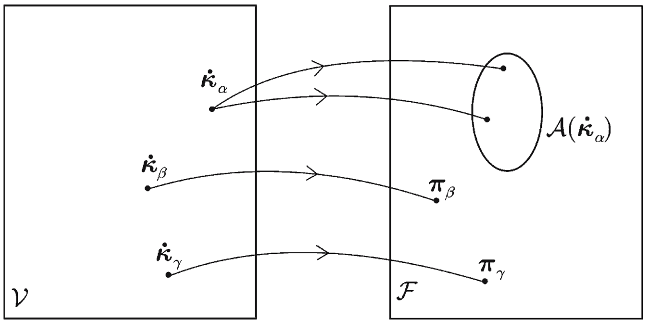

Figure 1. Dissipative mapping.

It is said that π and $\dot{\kappa}$ are conjugated with respect to the dissipation. While the evolution of the strain can be controlled externally, the internal variables evolve according to some additional laws called *evolution laws* which complement the state laws (*e.g.* elastic law for an elastic perfectly plastic model). These laws, which describe the evolution of the internal modifications, establish relationships between the rate of change of each κ_i and each generalized stress π_i. From a mathematical viewpoint, the global evolution law defines a certain mapping between $\mathcal{V}$ and $\mathcal{F}$, denoted by $\mathcal{A}$, which maps each $\dot{\kappa} \in \mathcal{V}$ to the set, possibly empty, $\mathcal{A}(\dot{\kappa}) \subset \mathcal{F}$ (see Figure 1).

The relationship between $\dot{\kappa}$ and π can be expressed in the following explicit form

$$\mathcal{A}: \mathcal{V} \to \mathcal{F}: \dot{\kappa} \mapsto \pi \in \mathcal{A}(\dot{\kappa}). \tag{2}$$

The operator $\mathcal{A}(\cdot)$ transforms $\dot{\kappa}$ defined in the domain $\mathscr{D}(\mathcal{A}) \subset \mathscr{V}$ into π defined in the range of the operator $\mathscr{R}(\mathcal{A}) \subset \mathcal{F}$. In the most general case, the dissipative mapping $\mathcal{A}$ will be non-linear. Further, the operator $\mathcal{A}$ is said to be single-valued or multi-valued at $\dot{\kappa}$ according to whether $\mathcal{A}(\dot{\kappa})$ is a singleton or a set containing more than one element (see Figure 1). The multi-valuedness is a desirable feature for a dissipative law like the flow rule in plasticity. In what follows, only invertible operators are considered. If the map $\mathcal{A}$ is invertible, the inverse law has the form:

$$\mathcal{A}^{-1}: \mathcal{F} \to \mathcal{V}: \pi \mapsto \dot{\kappa} \in \mathcal{A}^{-1}(\pi). \tag{3}$$

A class of dissipative materials, interesting from both a mathematical and a computational point of view, are those for which the dissipative operator can be obtained as a gradient or a subgradient[1] of a function for all its elements of its domain. For a single-valued operator, the condition ensuring that such a function does, in fact, exist, is the conservativity of the operator, *i.e.*, the vanishing of the related integral along every closed curve in the domain of the operator. If the constitutive operator is differentiable, this condition is ensured by the symmetry of its first Fréchet derivative D

$$[D\mathcal{A}(\dot{\kappa})\mathrm{d}\dot{\kappa}]\delta\dot{\kappa} = [D\mathcal{A}(\dot{\kappa})\delta\dot{\kappa}]\mathrm{d}\dot{\kappa} \tag{4}$$

for any vectors $\mathrm{d}\dot{\kappa}$ and $\delta\dot{\kappa}$ in $\mathcal{V}$. In that case, the constitutive operator can be obtained as the gradient of a scalar-valued function $\phi(\dot{\kappa})$ for all its elements of its domain:

$$\pi = D\phi(\dot{\kappa}). \tag{5}$$

[1]The subgradient is a generalization of the gradient to non-differentiable functions.

In order for the potential $\phi(\dot{\boldsymbol{\kappa}})$ to be convex the operator $\mathcal{A}$ must be positive definite. The "potentiality" condition (4) and the convexity condition must be checked separately before we can claim that an operator derives from a convex potential. There is no complete answer as to whether a multi-valued operator corresponds to the subgradient operator of a scalar-valued function which is not necessarily convex. Some partial results exist which make use of the prox-regularity concept [7, pp. 609–618]. However, if the operator satisfies the maximal cyclic monotonicity condition, it is proven that this operator can be derived as the subgradient of a *convex* scalar-valued function. The operator $\mathcal{A}$ is *cyclically monotonic* if for any family of pairs $(\dot{\boldsymbol{\kappa}}_i, \boldsymbol{\pi}_i) \in \mathcal{V} \times \mathcal{F}$, $i = 0, 1, \ldots, n$ such that $\dot{\boldsymbol{\kappa}}_i \in \mathcal{A}^{-1}(\boldsymbol{\pi}_i)$, the following inequality holds

$$\sum_{i=0}^{n} (\boldsymbol{\pi}_{i+1} - \boldsymbol{\pi}_i) \cdot \dot{\boldsymbol{\kappa}}_i \leq 0, \quad \text{with } n+1 \equiv 0. \tag{6}$$

Therefore the relationship between $\dot{\boldsymbol{\kappa}}$ and $\boldsymbol{\pi}$ takes the following potential structure

$$\dot{\boldsymbol{\kappa}} \in \partial\phi(\boldsymbol{\pi}), \tag{7}$$

where $\phi(\boldsymbol{\pi})$ is a convex scalar-valued function satisfying

$$\phi(\boldsymbol{\pi}') - \phi(\boldsymbol{\pi}) \geq \dot{\boldsymbol{\kappa}} \cdot (\boldsymbol{\pi}' - \boldsymbol{\pi}). \tag{8}$$

The relation (7) represents a *differential inclusion* and the function $\phi(\boldsymbol{\pi})$ is called a *pseudo-potential* where the term "pseudo" is used to emphasize that this function is non-differentiable. The multi-valued character of the relationship $\dot{\boldsymbol{\kappa}}(\boldsymbol{\pi})$ lies in the non-differentiability of $\phi(\boldsymbol{\pi})$ which requires the use of the mathematical operator "$\in$". The condition (6) seems to be quite complicated to use in practice and it is preferred to consider two pairs $(\dot{\boldsymbol{\kappa}}_0, \boldsymbol{\pi}_0)$ and $(\dot{\boldsymbol{\kappa}}_1, \boldsymbol{\pi}_1)$ to obtains the inequality

$$(\boldsymbol{\pi}_0 - \boldsymbol{\pi}_1) \cdot (\dot{\boldsymbol{\kappa}}_0 - \dot{\boldsymbol{\kappa}}_1) \geq 0, \tag{9}$$

which means that the mapping is *monotonically increasing*. If we can find two pairs that violate inequality (9), then the mapping is not monotonically increasing. Obviously, if the mapping is not monotonically increasing, it is not cyclically monotonic. Practically, the condition (9) is used as an necessary test. To rule out the existence of a convex pseudo-potential, it is enough to show that there exist two pairs violating the inequality (9). If we assume the existence of a convex subset $K \subset \mathcal{F}$

$$K \subset \mathcal{F} = \{\boldsymbol{\pi} \in \mathcal{F} \mid f(\boldsymbol{\pi}) \leq 0\}, \tag{10}$$

such that $\dot{\boldsymbol{\kappa}} \neq 0$ if $f(\boldsymbol{\pi}) = 0$, the condition (9) is equivalent to the normality rule. Accordingly, if the evolution law does not satisfy the normality then there is no convex pseudo-potential. In the next section more details will be given about rate-independent models.

The primary advantage of having a potential structure for the constitutive relations is that both analytical and physical insights may be gained. At least for geometrically linear solid mechanics boundary-value problems, it will result in the possibility of applying the calculus of variations. This technique consists of replacing the problem with a system of differential or integro-differential equations (BVP) by the equivalent problem of seeking the stationary points of a proper functional. The extremal formulations make the qualitative study of the problem easier, *i.e.*, the study of the existence, uniqueness and regularity of the solution through the so-called direct methods of variational calculus.

A broad range of dissipative materials present in engineering have more complex dissipative laws which can not take the convenient form of a potential law. One of most illustrative example is the Coulomb frictional-contact law. Other examples are typically those provided by dissipative laws of geomaterials and cyclic (visco)-plasticity models. In this context, the following question arises naturally: how can one preserve all the benefits of a formulation based on the definition of a scalar-valued function, *i.e.*, a potential structure of the dissipative law? An answer to this question is to relax the explicit relation introduced by the potential form by admitting an implicit one.

3. Potential and pseudo-potential

In this section, fundamental aspects of the material modeling discussed in the introduction are illustrated by considering two classical models. The first one is the linear viscous model which under usual symmetry conditions leads to a pair of dual differentiable potentials. The second example is the rigid-perfectly-plastic model where the associated flow rule (with complementarity relations) defines a multi-valued operator that can be derived from a non-differentiable potential. Further details on potential in constitutive modeling are given by Mróz [8, pp. 1–37].

3.1. Potential

Among classical dissipative models, the linear viscous material provides probably the most elementary dissipative law. There is only one internal variable which corresponds to the viscous deformation $\boldsymbol{\varepsilon}^v$, conjugated to the Cauchy stress tensor $\boldsymbol{\sigma}$. The evolution law takes the following simple form

$$\mathcal{E} \mapsto \mathcal{S} : \dot{\boldsymbol{\varepsilon}}^v \mapsto \boldsymbol{\sigma} = \mathcal{L}\,\dot{\boldsymbol{\varepsilon}}^v, \tag{11}$$

where $\mathcal{L}$ is a linear mapping represented by a 6×6 matrix whose elements are constant. The spaces $\mathcal{E}$ and its dual $\mathcal{S}$ merely correspond the six-dimensional space of symmetric second-order tensors. We suppose that $\mathcal{L}$ is symmetric and invertible. Trivially, the scalar-valued function $\psi(\dot{\boldsymbol{\varepsilon}}^v)$

$$\psi : \mathcal{E} \mapsto \mathbb{R} : \dot{\boldsymbol{\varepsilon}}^v \mapsto \frac{1}{2}(\dot{\boldsymbol{\varepsilon}}^v)^{\mathrm{t}}\mathcal{L}\,\dot{\boldsymbol{\varepsilon}}^v \tag{12}$$

is a quadratic form on the velocity space $\mathcal{E}$ and $\mathcal{L}\,\dot{\boldsymbol{\varepsilon}}^v$ is its gradient at $\dot{\boldsymbol{\varepsilon}}^v$. The potential ψ is convex only if the operator $\mathcal{L}$ is positive definite

$$(\dot{\boldsymbol{\varepsilon}}^v)^{\mathrm{t}}\mathcal{L}\,\dot{\boldsymbol{\varepsilon}}^v \geq 0. \tag{13}$$

Thus, the relation (11) may be equivalently written as

$$\boldsymbol{\sigma} = \operatorname{grad}\psi(\dot{\boldsymbol{\varepsilon}}^v). \tag{14}$$

A nice consequence of the *normality rule* (14) is the possibility to make use of the Legendre transform to invert the law (11). If $\boldsymbol{\sigma}$ is related to $\dot{\boldsymbol{\varepsilon}}^v$ by means of a potential $\psi(\dot{\boldsymbol{\varepsilon}}^v)$, Legendre showed that $\dot{\boldsymbol{\varepsilon}}^v$ is, in turn, related to $\boldsymbol{\sigma}$ through a potential ψ^* such that

$$\dot{\boldsymbol{\varepsilon}}^v = \operatorname{grad}\psi^*(\boldsymbol{\sigma}). \tag{15}$$

The potential $\psi^*(\boldsymbol{\sigma})$ is equal to

$$\psi^*(\boldsymbol{\sigma}) = \dot{\boldsymbol{\varepsilon}}^v \cdot \operatorname{grad}\psi(\dot{\boldsymbol{\varepsilon}}^v) - \psi(\dot{\boldsymbol{\varepsilon}}^v), \tag{16}$$

and its expression in terms of $\boldsymbol{\sigma}$ is

$$\psi^*(\boldsymbol{\sigma}) = (\mathcal{L}^{-1}\boldsymbol{\sigma}) \cdot \operatorname{grad} \psi(\mathcal{L}^{-1}\boldsymbol{\sigma}) - \psi(\mathcal{L}^{-1}\boldsymbol{\sigma}). \tag{17}$$

If the operator $\mathcal{L}$ is linear then ψ^* is given by

$$\psi^* : \mathcal{S} \mapsto \mathbb{R} : \boldsymbol{\sigma} \mapsto \frac{1}{2}\boldsymbol{\sigma}^{\mathrm{t}}\mathcal{L}^{-1}\boldsymbol{\sigma}. \tag{18}$$

The functions $\psi(\dot{\boldsymbol{\varepsilon}}^v)$ and $\psi^*(\boldsymbol{\sigma})$ are conjugate (or dual) and related by the following equality

$$\psi(\dot{\boldsymbol{\varepsilon}}^v) + \psi^*(\boldsymbol{\sigma}) = \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}^v. \tag{19}$$

For any pair $(\dot{\boldsymbol{\varepsilon}}^{v\prime}, \boldsymbol{\sigma}') \in \mathcal{E} \times \mathcal{S}$ not related by the constitutive relation, we have

$$\psi(\dot{\boldsymbol{\varepsilon}}^{v\prime}) + \psi^*(\boldsymbol{\sigma}') \neq \boldsymbol{\sigma}' \cdot \dot{\boldsymbol{\varepsilon}}^{v\prime}. \tag{20}$$

It is worth mentioning that the Legendre transform does not require that the potential $\psi(\dot{\boldsymbol{\varepsilon}}^v)$ is convex. However, if the constitutive operator satisfies (13), the potential $\psi(\dot{\boldsymbol{\varepsilon}}^v)$ is convex and $\psi^*(\boldsymbol{\sigma})$, which is also convex, can be obtained using the following maximization procedure

$$\psi^*(\boldsymbol{\sigma}) = \sup_{\dot{\boldsymbol{\varepsilon}}^v} [\boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}^v - \psi(\dot{\boldsymbol{\varepsilon}}^v)]. \tag{21}$$

When ψ is concave, the same formula is used where "sup" is replaced by a "inf". We can remark that, with these definitions, there is no need for the potential to be differentiable. As a consequence of (21), we have

$$\psi(\dot{\boldsymbol{\varepsilon}}^{v\prime}) + \psi^*(\boldsymbol{\sigma}') \geq \boldsymbol{\sigma}' \cdot \dot{\boldsymbol{\varepsilon}}^{v\prime}, \quad \forall (\dot{\boldsymbol{\varepsilon}}^{v\prime}, \boldsymbol{\sigma}') \in \mathcal{E} \times \mathcal{S}. \tag{22}$$

The mathematical properties of the potentials reflect the nature of the behavior. The one-to-one relation is related to the differentiability of the potential. The convexity of the potentials is a consequence of the monotonic nature of the behavior. Finally, quadratic form implies a linear relation between static and kinematic quantities. All information about the behavior is contained in the function $\psi(\dot{\boldsymbol{\varepsilon}}^v)$. Probably the most attractive property is the existence of variational principles. Weak formulations of a boundary-value problem involving such materials lead to differentiable minimum principles. It is found that the term variational may be understood in different ways. This term can mean: weak formulation (*i.e.*, principle of virtual work), stationary principle or extremum principle. Although a weak formulation can be associated with most of the physical problems, a few of them admit a stationary principle and even fewer admit an extremum principle. Convex or concave potentials lead to extremum principles which are particularly attractive from both a mathematical and a computational point of view. In fact, mathematicians have used them to prove existence and eventually uniqueness of the solution to the corresponding boundary-value problem. Further, the possibility of searching for the solution of a physical problem as a minimum point of a convex functional on a convex set is especially relevant from a computational point view.

Consider a perfectly viscous body occupying a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary Γ, subjected to imposed body force $\bar{\boldsymbol{b}}$, imposed surface traction $\bar{\boldsymbol{t}}$ on the part Γ_t of Γ, and imposed velocity $\bar{\dot{\boldsymbol{u}}}$ on the remaining part of the boundary $\Gamma_u = \Gamma - \Gamma_t$. We assume that the body is fixed on Γ_u, *i.e.*, $\bar{\dot{\boldsymbol{u}}} = \boldsymbol{0}$. Velocities and strain rates are assumed to be small, so that geometry changes can be neglected and the analysis is performed on the reference configuration. The weak form of equilibrium equations yields to the following functional

$$\int_{\Omega} \boldsymbol{\sigma} \cdot \delta\dot{\boldsymbol{\varepsilon}}^{v}\, \mathrm{d}\Omega - \int_{\Omega} \bar{\boldsymbol{b}} \cdot \delta\dot{\boldsymbol{u}}\, \mathrm{d}\Omega - \int_{\Gamma_t} \bar{\boldsymbol{t}} \cdot \delta\dot{\boldsymbol{u}}\, \mathrm{d}\Gamma = 0, \tag{23}$$

where $\delta\dot{\boldsymbol{u}} \in \mathscr{V}_{ad}$ is defined by

$$\mathscr{V}_{ad} = \left\{ \delta\dot{\boldsymbol{u}} \in \mathscr{V} \mid \delta\dot{\boldsymbol{\varepsilon}}^{v} = \nabla_S\, \delta\dot{\boldsymbol{u}} \quad \text{in}, \Omega\,, \; \delta\dot{\boldsymbol{u}} = \mathbf{0} \quad \text{on}\, \Gamma_u \right\};$$

here ∇_S is the symmetric part of the gradient operator. As a result of the potential structure of the constitutive relation, a solution of the virtual work equation (23) corresponds to a stationary point

$$\mathscr{J}(\dot{\boldsymbol{u}}) \rightarrow \text{stationary} \quad \text{over} \quad \mathscr{V}_{ad}, \tag{24}$$

where

$$\mathscr{J}(\dot{\boldsymbol{u}}) = \int_{\Omega} \psi(\dot{\boldsymbol{\varepsilon}}^{v}(\dot{\boldsymbol{u}}))\, \mathrm{d}\Omega - \int_{\Omega} \bar{\boldsymbol{b}} \cdot \dot{\boldsymbol{u}}\, \mathrm{d}\Omega - \int_{\Gamma_t} \bar{\boldsymbol{t}} \cdot \dot{\boldsymbol{u}}\, \mathrm{d}\Gamma, \tag{25}$$

which is analogous to the energy functional for elastic bodies. The functional $\mathrm{J}(\dot{\boldsymbol{u}})$ is convex if $\psi(\dot{\boldsymbol{\varepsilon}}^{v})$ is convex/concave and the stationary principle becomes a minimum/maximum principle:

$$\mathscr{J}(\dot{\boldsymbol{u}}) \longrightarrow \min \quad \text{over} \quad \mathscr{V}. \tag{26}$$

In solid mechanics, the weak form (23) always exists but is equivalent to (24) only if the symmetry conditions are satisfied. The second advantage of having a potential is that properties of the solutions appear more explicitly.

Indeed, if J is convex/concave the BVP (functional) has a unique solution but if $\mathscr{J}$ is non-convex it has more than one solution (Figure 2). The extremal formulations are particularly suitable for finding numerical solutions of the problem through direct solution procedures based on combining finite-element and optimization procedures.

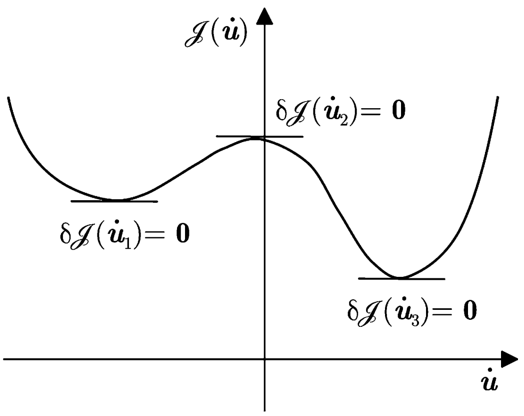

Figure 2. Non-convex functional.

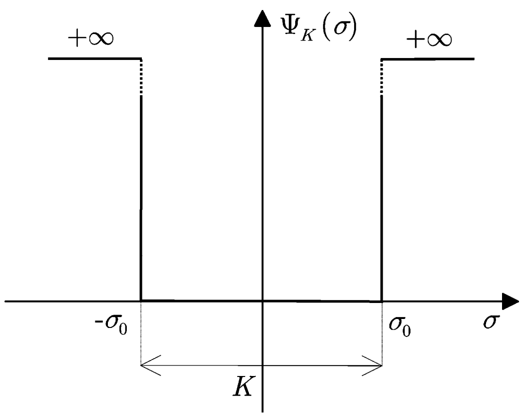

Figure 3. Indicator function for the uniaxial plastic model.

3.2. Pseudo-potential

The concept of potential, although attractive, is not relevant to describe all dissipative laws. The most simple counter-example is given by the classical Mises metal plasticity. The internal variable is the plastic strain tensor $\boldsymbol{\varepsilon}^p \in \mathcal{E}$ and the corresponding associated variable is the Cauchy stress tensor $\boldsymbol{\sigma} \in \mathcal{S}$. For convenience, the plastic strain is decomposed into the volumetric plastic strain e_m^p (belonging to the one-dimensional vector space $\mathcal{E}_m$) and the plastic strain deviator $\mathbf{e}^p$ (belonging to the five-dimensional vector space $\mathcal{E}_d$). The corresponding dual variables are the mean stress s_m (belonging to the space $\mathcal{S}_m$, dual of $\mathcal{E}_m$) and the stress deviator $\mathbf{s}$ (belonging to the space $\mathcal{S}_d$, dual of $\mathcal{E}_d$). Clearly, we have the following decomposition

$$\mathcal{E} = \mathcal{E}_m \oplus \mathcal{E}_d \quad \text{and} \quad \mathcal{S} = \mathcal{S}_m \oplus \mathcal{S}_d$$

and the dual pairing is defined by the bilinear form:

$$\dot{\boldsymbol{\varepsilon}}^p \cdot \boldsymbol{\sigma} = \dot{e}_m^p s_m + \dot{\mathbf{e}}^p \cdot \mathbf{s}. \tag{27}$$

The space $\mathcal{S}_d$ is equipped with the von Mises norm

$$\|\mathbf{s}\|_{eq} = \left(\frac{3}{2}\|\mathbf{s}\|^2\right)^{\frac{1}{2}}, \tag{28}$$

which is used to define a closed convex set of admissible stresses, denoted K

$$K = \left\{(s_m, \mathbf{s}) \in \mathcal{S} \mid \|\mathbf{s}\|_{eq} - \sigma_0 \le 0\right\}, \tag{29}$$

where σ_0 is the yield stress. For an associated flow rule, the direction of the plastic strain rate is given by the gradient to the yield function and its magnitude by the plastic multiplier:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}}. \tag{30}$$

The latter is required to satisfy the complementarity relations

$$f(\boldsymbol{\sigma}) \le 0, \quad \dot{\lambda} \ge 0, \quad \dot{\lambda} f(\boldsymbol{\sigma}) = 0. \tag{31}$$

The previous relations refer to the rate formulation of the plastic flow and is probably the most popular form of the flow rule. To gain more insight into the nature of the plastic behavior, the complete flow rule can be written analytically, using a '*if ... then ... else*' statement, as follows

If $\boldsymbol{\sigma} \in \operatorname{int} K$ *then*

$\dot{\boldsymbol{\varepsilon}}^p = \mathbf{0}$! elastic loading/unloading

else

$\left\{\boldsymbol{\sigma} \in \operatorname{bd} K \text{ and } \exists\, \dot{\lambda} > 0 \text{ such that } \dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\}$! plastic loading

endif

where "int K" and "bd K" denotes the interior and the boundary of K, repectively. The first part of the statement highlights an important feature of the flow rule, namely its multi-valued nature. Indeed, the zero plastic strain rate can be related to an infinite number of stress states which correspond to the whole elastic domain. As a consequence the constitutive operator $\mathcal{A}$

is multi-valued and therefore it cannot be obtained as a gradient of a differentiable potential. We will see that a potential form of the flow rule can still be derived but by considering a non-differentiable potential, the non-differentiability being required by the multi-valued character of the flow rule. An alternative form of the flow rule (30) and the complementarity relations (31) is the one given by the maximum dissipation inequality, known as Hill's principle:

$$\boldsymbol{\sigma} \in K : \dot{\boldsymbol{\varepsilon}}^p \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}') \geq 0, \quad \forall \boldsymbol{\sigma}' \in K. \tag{32}$$

The variational inequality is appreciated by mathematicians because it is a suitable tool for proving the existence of solutions. By transforming adequately the inequality (32), one can obtain the set-valued mapping relating the stress $\boldsymbol{\sigma}$ and the plastic strain rate $\dot{\boldsymbol{\varepsilon}}^p$, *i.e.,* the relationship $\dot{\boldsymbol{\varepsilon}}^p(\boldsymbol{\sigma})$. The idea, due to Moreau [2,3], is to make use of the indicator function [7, p. 6] of the set K to which the stresses $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ are required to belong. This particular function, frequently used in Convex Analysis, is defined by

$$\Psi_K(\boldsymbol{\sigma}) = \begin{cases} 0 & \text{if } \boldsymbol{\sigma} \in K, \\ +\infty & \text{otherwise.} \end{cases} \tag{33}$$

This function has a zero ground level within the elastic domain and infinite walls along the yield surface. Figure 3 gives a schematic interpretation of the indicator function for the one-dimensional case.

The function $\Psi_K(\boldsymbol{\sigma})$ is not differentiable in the classical sense. However, the indicator function is convex if the set to which it refers is convex. Having at hand this tool, a key step is to rewrite the variational inequality (32) in the following manner

$$\dot{\boldsymbol{\varepsilon}}^p \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}') + \Psi_K(\boldsymbol{\sigma}') \geq \Psi_K(\boldsymbol{\sigma}), \tag{34}$$

where the member function "$\in$" in (32) has been replaced by the value of the indicator function at the corresponding stress. Both inequalities (32) and (34) are equivalent. In fact, we remark that, if inequality (32) is satisfied, it follows that inequality (34) is also satisfied. Conversely, if inequality (34) holds, by taking $\boldsymbol{\sigma}'$ in K, we see that $\Psi_K(\boldsymbol{\sigma}')$ has a finite value (zero) and $\Psi_K(\boldsymbol{\sigma})$ must be equal to zero which means that $\boldsymbol{\sigma}$ is in K and therefore inequality (32) is fulfilled. The inequality (34) corresponds to the *convexity inequality* applied to a non-differentiable function [7, p. 301]. It means that $\dot{\boldsymbol{\varepsilon}}^p$ belongs to the subdifferential of $\Psi_K(\boldsymbol{\sigma})$ at $\boldsymbol{\sigma}$ or equivalently, $\dot{\boldsymbol{\varepsilon}}^p$ and $\boldsymbol{\sigma}$ are related by the differential inclusion:

$$\dot{\boldsymbol{\varepsilon}}^p \in \partial \Psi_K(\boldsymbol{\sigma}). \tag{35}$$

The subdifferential of $\Psi_K(\boldsymbol{\sigma})$ at $\boldsymbol{\sigma}$ corresponds to the set of all subgradients of $\Psi_K(\boldsymbol{\sigma})$ at $\boldsymbol{\sigma}$. It defines a multi-valued operator that maps each point in the domain $\mathcal{S}$ of the function to the closed convex set of its subgradient:

$$\partial \Psi_K(\boldsymbol{\sigma}) : \mathcal{S} \mapsto \mathcal{E} : \boldsymbol{\sigma} \mapsto \partial \Psi_K(\boldsymbol{\sigma}). \tag{36}$$

In particular, for a differentiable function, the subdifferential is reduced to a singleton which corresponds to the classical gradient. The function $\Psi_K(\boldsymbol{\sigma})$ is called complementary dissipation pseudo-potential and denoted by $\psi^*(\boldsymbol{\sigma})$. The relation (35) is equivalent to the flow rule and the complementarity conditions. The above considerations show that by simply allowing the potential to be non-differentiable, we can produce a "potential structure" of the relationship between $\dot{\boldsymbol{\varepsilon}}^p$ and $\boldsymbol{\sigma}$. Accordingly, we can say that differentiable potentials are suited only for single-valued laws and non-differentiable potentials provide an effective means of representing multi-valued constitutive laws. With the setting (35), the relationship may be inverted by applying the Fenchel transform

$$\psi(\dot{\boldsymbol{\varepsilon}}^p) = \sup_{\boldsymbol{\sigma}} \left[\boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}^p - \Psi_K(\boldsymbol{\sigma})\right] = \sup_{\boldsymbol{\sigma} \in K} \left[\boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}^p\right], \tag{37}$$

where $\psi(\dot{\boldsymbol{\varepsilon}}^p)$ is the dissipation pseudo-potential, which is convex by construction. The inverse flow rule is then

$$\boldsymbol{\sigma} \in \partial\psi(\dot{\boldsymbol{\varepsilon}}^p), \tag{38}$$

which is equivalent to

$$\boldsymbol{\sigma} \cdot (\dot{\boldsymbol{\varepsilon}}^p - \dot{\boldsymbol{\varepsilon}}^{p\prime}) + \psi(\dot{\boldsymbol{\varepsilon}}^{p\prime}) \geq \psi(\dot{\boldsymbol{\varepsilon}}^p). \tag{39}$$

The formulations (35) and (38) of the flow rule and its inverse are particularly useful for associating dual extremum principles to boundary-value problems involving rigid plastic materials with associated flow rules. The dissipated power is exactly equal to $\psi(\dot{\boldsymbol{\varepsilon}}^p)$; hence, in the present case, $\psi(\dot{\boldsymbol{\varepsilon}}^p)$ may be called the dissipation function of the material. The previous developments are illustrated by considering the deviatoric plastic model. The expression of $\psi^*(\boldsymbol{\sigma})$ does not pose any particular difficulty as it always coincides with the indicator function of the elastic domain but we need to derive the expression of the dissipation pseudo-potential. Using the decomposition (27), the scalar product can be decomposed as

$$\sup_{\boldsymbol{\sigma} \in K} \left[\boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}^p\right] = \sup_{\boldsymbol{\sigma} \in K} \left[\mathbf{s} \cdot \dot{\mathbf{e}}^p + s_m \, \dot{e}_m^p\right]. \tag{40}$$

It is clear that the supremum will be achieved for a vector $\boldsymbol{\sigma}$ colinear to $\dot{\boldsymbol{\varepsilon}}^p$:

$$\mathbf{s} \cdot \dot{\mathbf{e}}^p + s_m \, \dot{e}_m^p \leq \|\mathbf{s}\| \, \|\dot{\mathbf{e}}^p\| + s_m \, \dot{e}_m^p. \tag{41}$$

To be able to use the yield criterion, we replace the Euclidean norm of $\mathbf{s}$ by the von Mises norm (28)

$$\mathbf{s} \cdot \dot{\mathbf{e}}^p + s_m \, \dot{e}_m^p \leq \|\mathbf{s}\|_{eq} \, \|\dot{\mathbf{e}}^p\|_{eq}^* + s_m \, \dot{e}_m^p, \tag{42}$$

where $\| \bullet \|_{eq}^*$ corresponds to the dual norm of $\| \bullet \|_{eq}$ and is defined on the space $\mathcal{E}_d$

$$\|\dot{\mathbf{e}}^p\|_{eq}^* = \left(\frac{2}{3}\|\dot{\mathbf{e}}^p\|^2\right)^{\frac{1}{2}}. \tag{43}$$

Taking into account that $\|\mathbf{s}\|_{eq}$ is bounded by σ_0, we have

$$\mathbf{s} \cdot \dot{\mathbf{e}}^p + s_m \, \dot{e}_m^p \leq \sigma_0 \, \|\dot{\mathbf{e}}^p\|_{eq}^* + s_m \, \dot{e}_m^p. \tag{44}$$

Two distinct possibilities emerge: if $\dot{e}_m^p = 0$, the supremum (40) is $\sigma_0 \, \|\dot{\mathbf{e}}^p\|_{eq}^*$. If on the other hand, $\dot{e}_m^p \neq 0$ then, since the value of s_m is unbounded, so is the supremum (40). Thus, the expression of the dissipation pseudo-potential is given by

$$\psi(\dot{\boldsymbol{\varepsilon}}^p) = \begin{cases} \sigma_0 \, \|\dot{\mathbf{e}}^p\|_{eq}^* & \text{if } \dot{e}_m^p = 0 \\ +\infty & \text{if } \dot{e}_m^p \neq 0 \end{cases}. \tag{45}$$

Introducing the indicator function $\Psi_{\{0\}}(\dot{e}_m^p)$ defined by

$$\Psi_{\{0\}}(\dot{e}_m^p) = \begin{cases} 0 & \text{if } \dot{e}_m^p = 0 \\ +\infty & \text{if } \dot{e}_m^p \neq 0 \end{cases}, \tag{46}$$

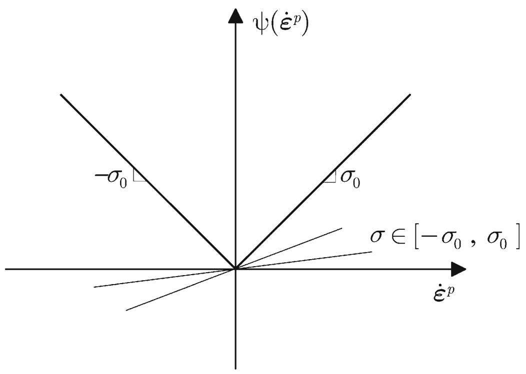

Figure 4. One-dimensional dissipation pseudo-potential.

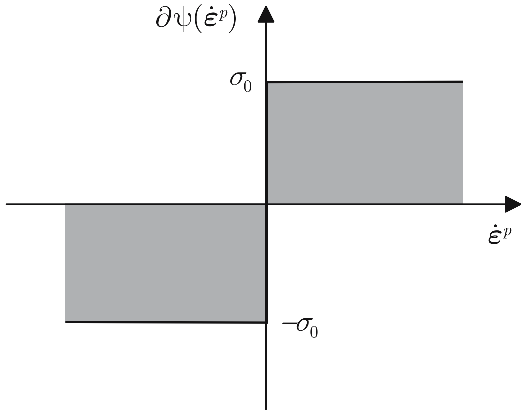

Figure 5. Graph of the rigid-plastic law.

we may write the expression of the dissipation pseudo-potential as

$$\psi(\dot{\boldsymbol{\varepsilon}}^p) = \sigma_0 \, \|\dot{\mathbf{e}}^p\|^*_{eq} + \Psi_{\{0\}}(\dot{e}^p_m). \tag{47}$$

As a consequence of the Fenchel transform property, the pseudo-potential of dissipation is convex. Moreover, it is a positive function, homogeneous of degree one which takes a zero value at the origin where the function is non-differentiable. In most papers, the dissipation function is defined without taking into account the internal constraints on the plastic strain rate (or internal variable rate in more general models). We decide to take into account these internal constraints by adding to the mechanical dissipation an indicator function, so that the dissipation has the value $+\infty$ for non-physical states. We call this function the "extended dissipation pseudo-potential". As a result, the dissipation is defined for any plastic strain rate vector $\dot{\boldsymbol{\varepsilon}}^p \in \mathcal{E}$ including those which are actually physically impossible. In one-dimension, the expression of the dissipation pseudo-potential (see Figure 4) becomes

$$\psi(\dot{\varepsilon}^p) = \sigma_0 \, |\dot{\varepsilon}^p|. \tag{48}$$

The pseudo-potential of dissipation is differentiable everywhere except at the origin (see on the graph). The subdifferential of $\psi(\dot{\varepsilon}^p)$ is given by

$$\partial\psi(\dot{\varepsilon}^p) = \begin{cases} -\sigma_0, & \sigma < 0 \\ (-\sigma_0, \sigma_0), & \sigma = 0 \\ \sigma_0, & \sigma > 0 \end{cases}. \tag{49}$$

It corresponds to the multi-valued rigid plastic model, geometrically represented in Figure 5. As can be seen from Figure 4, the subdifferential at the origin corresponds the elastic domain.

$$K = \partial\psi(\mathbf{0}).$$

Again, the set-valued relation is related to the non-differentiability of the potential and the convexity of the potentials is a consequence of the monotonic nature of the behavior. Finally, the fact that the pseudo-potential is homogeneous of degree one implies a rate-independent behavior. All relevant information about the behavior is contained in the function $\psi(\dot{\boldsymbol{\varepsilon}}^p)$. Furthermore, the convexity of the pseudo-potential of dissipation is a consequence of the maximum of dissipation principle. The functions $\psi(\dot{\boldsymbol{\varepsilon}}^p)$ and $\psi^*(\boldsymbol{\sigma})$ satisfy the following relation:

$$\psi(\dot{\boldsymbol{\varepsilon}}^{p\prime})+\psi^*(\boldsymbol{\sigma}')\geq\dot{\boldsymbol{\varepsilon}}^{p\prime}\cdot\boldsymbol{\sigma}',\quad\forall\left(\dot{\boldsymbol{\varepsilon}}^{p\prime},\boldsymbol{\sigma}'\right)\in\mathcal{E}\times\mathcal{S}.\tag{50}$$

Equality is reached when a couple $(\dot{\boldsymbol{\varepsilon}}^p,\boldsymbol{\sigma})$ is linked by the evolution law. Finally, the evolution law can be expressed equivalently by

$$\dot{\boldsymbol{\varepsilon}}^p\in\partial\Psi_K(\boldsymbol{\sigma})\Leftrightarrow\boldsymbol{\sigma}\in\partial\psi(\dot{\boldsymbol{\varepsilon}}^p)\Leftrightarrow\psi(\dot{\boldsymbol{\varepsilon}}^p)+\psi^*(\boldsymbol{\sigma})=\dot{\boldsymbol{\varepsilon}}^p\cdot\boldsymbol{\sigma}.\tag{51}$$

Although the pseudo-potential was introduced here for the simple model of rigid-perfect plasticity, the previous approach can be easily extended to take into account hardening effects by providing additional (to the plastic strain) internal variables. The elastic domain is then expressed in the generalized stress space and the rate form of the evolution law takes the form of a generalized normality rule. This leads to the class of Generalized Standard Materials (GSM) of Bernard Halphen and Nguyen Quoc Son [4, 5] who encompass a large class of materials (plasticity, viscoplasticity and damage models). An abundant literature can be found about evolution laws of plastic materials. In particular, contributions have been devoted to the analysis of the connections existing among Hill's principle of maximum dissipation, the existence of a pseudo-potential of dissipation and the normality rule to a convex elastic locus (see [9, pp. 71–83]).

4. The Cam-clay model

It is well known that geomaterials have a very complicated behavior compared to metals, even if only monotonic loading is considered. It is therefore a challenge in geomechanics to develop relatively simple mathematical models that are able to predict, at least qualitatively, a great number of fundamental aspects of soil behavior. The success of the modified Cam-clay model lies in its ability to capture many of the characteristics of clay behavior by using only a limited number of material parameters. This model belongs to the class of critical-state models which originated from the work of Roscoe and his co-workers at the University of Cambridge [6]. Recent work on the modified Cam-clay model using thermodynamics has been carried out by Collins [10]. Commonly observed features such as hardening/softening, contractancy/dilatancy and the tendency to eventually reach a state in which the stress state and the volume change become stationary are all captured by the modified Cam-clay model. Even at present, the modified Cam-clay model remains widely used for computational applications as further evidence of its success. In this section, we first recall the relations governing the dissipation of the model. The elasticity relations are not discussed (Figure 6).

5. Classical formulation

The modified Cam-clay yield surface (see Figure 6) is defined by

$$f(\mathbf{s},s_m,p_c)=\|\mathbf{s}\|_{eq}^2+M^2s_m^2-2\,M^2s_m\,p_c,\tag{52}$$

where p_c is the "*critical state pressure*" and M a material constant defined by

$$M=\frac{6\sin\phi}{3-\sin\phi}$$

with ϕ the internal friction angle. The solid-mechanics convention of strains and stresses is used. In the plane $(s_m,\|\mathbf{s}\|_{eq})$, the yield surface is represented by a family of ellipses (Figure 6) passing through the origin, taking a maximum value for $s_m=p_c$ and intercepting the mean

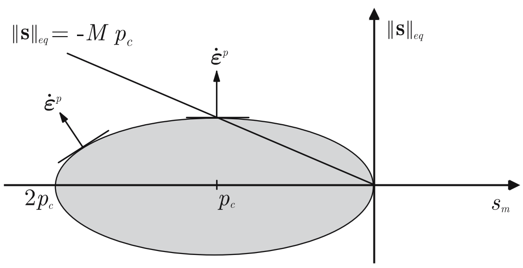

Figure 6. The Cam-clay Model.

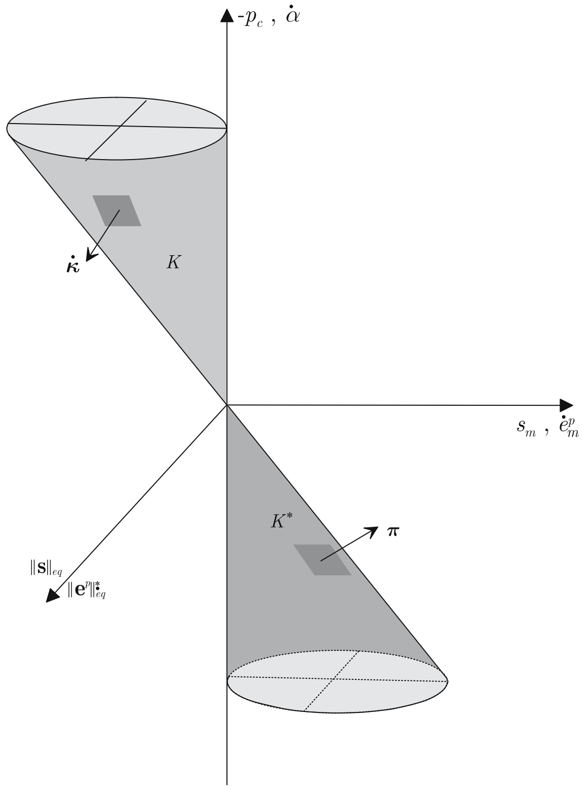

Figure 7. Modified Cam-clay in the generalized stress space.

stress axis at $s_m = 2\,p_c$. This point corresponds to the elastic limit under a hydrostatic loading and is called the "*preconsolidation pressure*". The plastic flow obeys the normality rule

$$\dot{\mathbf{e}}^p = \dot{\lambda}\frac{\partial f}{\partial \mathbf{s}} = 3\,\dot{\lambda}\,\mathbf{s}, \tag{53}$$

$$\dot{e}^p_m = \dot{\lambda}\frac{\partial f}{\partial s_m} = 2\,\dot{\lambda}\,M^2(s_m - p_c) \tag{54}$$

and the evolution of the elastic domain is governed by the relation

$$\dot{p}_c = \upsilon\, p_c\, \dot{e}^p_m \tag{55}$$

with

$$\upsilon = \frac{1+e}{\eta - \varsigma},$$

where e is the void ratio of the soil mass, η is the virgin compression index and ς the swell/re-compression index. Equation (55) shows that the contractancy leads to a decrease of p_c and therefore the ellipse expands so that the elastic domain is enlarged. On the contrary, dilatancy leads to an increase of p_c (softening phase) which corresponds to a reduction of the elastic domain. When the plastic volumetric strain is zero, p_c becomes constant and the elastic domain stationary. During hardening/softening, the top of the ellipse moves along the straight line $\|\mathbf{s}\|_{eq} = -M\,p_c$ called the "*critical state line*". The critical state line divides the stress space into a contractant and a dilatant region. For a stress state situated on the part of the yield function where $s_m > p_c$, the behavior will be plastically dilatant. On the contrary, for any stress state situated on the part of the yield function where $s_m < p_c$, the behavior will be plastically contractant. The point situated on the yield curve at the intersection with the critical-state line strains at constant plastic volume.

6. Internal-variable formulation

As mentioned before, our aim is to discuss the dissipative behavior of the model. In addition to the plastic strain $\dot{\boldsymbol{\varepsilon}}^p$, a scalar internal variable α is introduced. This variable accounts for hardening/softening. The conjugated variables are $\boldsymbol{\sigma}$ and p_c, respectively, and the dissipation is given by

$$\boldsymbol{\pi} \cdot \dot{\boldsymbol{\kappa}} = \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}^p - p_c\, \dot{\alpha} = \dot{\mathbf{e}}^p \cdot \mathbf{s} + \dot{e}_m^p\, s_m - p_c\, \dot{\alpha}, \tag{56}$$

where $\boldsymbol{\pi}$ and $\dot{\boldsymbol{\kappa}}$ are given by

$$\boldsymbol{\pi} = \begin{bmatrix} \boldsymbol{\sigma} \\ p_c \end{bmatrix} \quad \text{and} \quad \dot{\boldsymbol{\kappa}} = \begin{bmatrix} \dot{\boldsymbol{\varepsilon}}^p \\ -\dot{\alpha} \end{bmatrix}. \tag{57}$$

The elastic domain K is now defined in the generalized stress space

$$K = \left\{ \boldsymbol{\pi} \in \mathcal{S} \mid \|\mathbf{s}\|_{eq}^2 + M^2 s_m^2 - 2\, M^2 s_m\, p_c \leq 0 \right\}. \tag{58}$$

This corresponds to a cone as shown in Figure 7. Relation (55), which is referred to as the rate form of the state equation, becomes

$$\dot{p}_c = \upsilon\, p_c\, \dot{\alpha}. \tag{59}$$

Comparing (55) and (59), we deduce the so-called hardening rule

$$\dot{e}_m^p - \dot{\alpha} = 0. \tag{60}$$

The evolution law for α does not satisfy the normal rule since we have

$$\dot{\lambda} \frac{\partial f}{\partial p_c} = -2\, \dot{\lambda}\, M^2 s_m \neq -\dot{\alpha}. \tag{61}$$

Therefore, the model is not standard generalized, since we do not have generalized normality (normality for each component of $\dot{\boldsymbol{\kappa}}$). Therefore, we need to introduce another scalar function, called plastic potential, from which the evolution laws can be deduced by applying the normality rule. The expression of the plastic potential is

$$g(\mathbf{s}, s_m, p_c) = \|\mathbf{s}\|_{eq}^2 + M^2 (p_c - s_m) \tag{62}$$

and the evolution laws are given by

$$\dot{\mathbf{e}}^p = \dot{\lambda} \frac{\partial g}{\partial \mathbf{s}} = 3\, \dot{\lambda}\, \mathbf{s}, \quad \dot{e}_m^p = \dot{\lambda} \frac{\partial g}{\partial s_m} = 2\, \dot{\lambda}\, M^2 (s_m - p_c), \quad -\dot{\alpha} = \dot{\lambda} \frac{\partial g}{\partial p_c} = 2\, \dot{\lambda}\, M^2 (p_c - s_m).$$

Before continuing, let us derive an equivalent expression for the yield function which will have the property of being homogeneous of degree one. The square in (52) is completed to get

$$\|\mathbf{s}\|_{eq}^2 + M^2 (s_m - p_c)^2 \leq M^2 p_c^2. \tag{63}$$

Taking into account that p_c is always negative, we obtain an alternative expression of the elastic domain given by

$$\sqrt{\|\mathbf{s}\|_{eq}^2 + M^2(s_m - p_c)^2} \leq -M\, p_c \tag{64}$$

and the expression of the yield function is now

$$f(\boldsymbol{\sigma}, p_c) = \sqrt{\|\mathbf{s}\|_{eq}^2 + M^2(s_m - p_c)^2} + M\, p_c, \tag{65}$$

which is homogenous of degree one:

$$f(\beta\, \boldsymbol{\sigma}, \beta\, p_c) = \beta\, f(\boldsymbol{\sigma}, p_c). \tag{66}$$

By introducing the following notations

$$\boldsymbol{\sigma} = \begin{bmatrix} \mathbf{s} \\ s_m \end{bmatrix} \quad \text{and} \quad \mathbf{X} = \begin{bmatrix} \mathbf{0} \\ p_c \end{bmatrix},$$

we obtain the yield function (65) as follows:

$$f(\boldsymbol{\sigma}, p_c) = \|\boldsymbol{\sigma} - \mathbf{X}\|_{cc} + M\, p_c,$$

where

$$\|\boldsymbol{\sigma} - \mathbf{X}\|_{cc} = \left(\|\mathbf{s}\|_{eq}^2 + M^2(s_m - p_c)^2\right)^{\frac{1}{2}}. \tag{67}$$

With this new expression of the yield function, the flow rule becomes

$$\dot{\mathbf{e}}^p = \frac{3\,\dot{\lambda}}{2}\, \frac{\mathbf{s}}{\|\boldsymbol{\sigma} - \mathbf{X}\|_{cc}}, \tag{68}$$

$$\dot{e}_m^p = \dot{\lambda}\, M^2 \frac{(s_m - p_c)}{\|\boldsymbol{\sigma} - \mathbf{X}\|_{cc}}, \tag{69}$$

which leads to the following expression for the plastic multiplier $\dot{\lambda}$

$$\dot{\lambda} = \|\dot{\boldsymbol{\varepsilon}}^p\|_{cc}^* = \left(\|\dot{\mathbf{e}}^p\|_{eq}^2 + \frac{(\dot{e}_m^p)^2}{M^2}\right)^{\frac{1}{2}}, \tag{70}$$

where the norm $\|\bullet\|_{cc}^*$ defined on the velocity space is dual to the norm (67) in the sense that

$$(\boldsymbol{\sigma} - \mathbf{X}) \cdot \dot{\boldsymbol{\varepsilon}}^p \leq \|\boldsymbol{\sigma} - \mathbf{X}\|_{cc}\, \|\dot{\boldsymbol{\varepsilon}}^p\|_{cc}^*. \tag{71}$$

7. Standard version of the modified Cam-clay model

Suppose that we would like to have a generalized standard model. The hardening rule (60) must be different in order to satisfy the generalized normality. Indeed, by applying the normality rule for the internal variable α, we find the following relationship for the hardening rule

$$-\dot{\alpha} = \dot{\lambda}\frac{\partial f}{\partial p_c} = -\dot{e}_m^p + M\, \|\dot{\boldsymbol{\varepsilon}}^p\|_{cc}^*. \tag{72}$$

So, to have a standard model the relation, Equation (72) should be used instead of (60). It is worth mentioning that now $-\alpha$ is a non-decreasing variable and therefore softening cannot occur. With this expression of $-\alpha$, the evolution rule for all internal variables can be written in the following compact relation:

$$\dot{\boldsymbol{\kappa}} \in \partial\psi^*(\boldsymbol{\pi}), \tag{73}$$

where $\psi^*(\boldsymbol{\pi})$ is the complementary dissipation pseudo-potential which corresponds to the indicator function of the elastic domain K expressed in the generalized stress space

$$\Psi_K(\boldsymbol{\pi}) = \begin{cases} 0 & \text{if } \boldsymbol{\pi} \in K, \\ +\infty & \text{otherwise} \end{cases}. \tag{74}$$

A standard model for clay seems not to be appropriate, since it does not reproduce the softening behavior observed experimentally. Furthermore, the dissipation is always equal to zero. Indeed, the dissipation is obtained as follows

$$\psi(\dot{\boldsymbol{\kappa}}) = \sup_{\boldsymbol{\pi}\in K} [\boldsymbol{\pi} \cdot \dot{\boldsymbol{\kappa}}] = \sup_{\boldsymbol{\pi}\in K} \left[\mathbf{s} \cdot \dot{\mathbf{e}}^p + s_m\, \dot{e}_m^p - p_c\dot{\alpha}\right]. \tag{75}$$

It is clear that the supremum will be achieved for a vector π colinear to $\dot{\kappa}$:

$$\mathbf{s} \cdot \dot{\mathbf{e}}^p + s_m\, \dot{e}_m^p - p_c\dot{\alpha} \le \|\mathbf{s}\|\, \|\dot{\mathbf{e}}^p\| + s_m\, \dot{e}_m^p - p_c\dot{\alpha}. \tag{76}$$

Adding and subtracting $p_c\,\dot{e}_m^p$, we have

$$\mathbf{s} \cdot \dot{\mathbf{e}}^p + s_m\, \dot{e}_m^p - p_c\dot{\alpha} \le \|\mathbf{s}\|\, \|\dot{\mathbf{e}}^p\| + (s_m - p_c)\, \dot{e}_m^p + p_c(\dot{e}_m^p - \dot{\alpha}). \tag{77}$$

Now, we make use of the Cauchy–Schwarz inequality (71), to obtain

$$\mathbf{s} \cdot \dot{\mathbf{e}}^p + s_m\, \dot{e}_m^p - p_c\dot{\alpha} \le \|\boldsymbol{\sigma} - \mathbf{X}\|_{cc}\, \|\dot{\boldsymbol{\varepsilon}}^p\|_{cc}^* + p_c(\dot{e}_m^p - \dot{\alpha}). \tag{78}$$

Taking into account that $\|\boldsymbol{\sigma} - \mathbf{X}\|_{cc}$ is bounded by $-M\,p_c$, we have

$$\mathbf{s} \cdot \dot{\mathbf{e}}^p + s_m\, \dot{e}_m^p - p_c\dot{\alpha} \le -p_c\, \left(M\, \|\dot{\boldsymbol{\varepsilon}}^p\|_{cc}^* + \dot{\alpha} - \dot{e}_m^p\right). \tag{79}$$

Two distinct possibilities emerge. If we have

$$M\, \|\dot{\boldsymbol{\varepsilon}}^p\|_{cc}^* + \dot{\alpha} - \dot{e}_m^p \le 0, \tag{80}$$

then because $-p_c$ is positive, the dissipation is equal to zero. On the other hand, if we have

$$M\, \|\dot{\boldsymbol{\varepsilon}}^p\|_{cc}^* + \dot{\alpha} - \dot{e}_m^p \ge 0, \tag{81}$$

then, since the value of $-p_c$ is unbounded, so is the supremum (75). Thus, we have

$$\psi(\dot{\boldsymbol{\kappa}}) = \begin{cases} 0 & \text{if } M\, \|\dot{\boldsymbol{\varepsilon}}^p\|_{cc}^* + \dot{\alpha} - \dot{e}_m^p \le 0 \\ +\infty & \text{if } M\, \|\dot{\boldsymbol{\varepsilon}}^p\|_{cc}^* + \dot{\alpha} - \dot{e}_m^p \ge 0 \end{cases}, \tag{82}$$

in short

$$\psi(\dot{\boldsymbol{\kappa}}) = \Psi_{K^*}(\dot{\boldsymbol{\kappa}}), \tag{83}$$

where K^* is the cone dual to the cone K (see Figure 7) and defined by

$$K^* = \{\dot{\boldsymbol{\kappa}} \in \mathcal{V} \mid M\, \|\dot{\boldsymbol{\varepsilon}}^p\|_{cc}^* + \dot{\alpha} - \dot{e}_m^p \le 0\}. \tag{84}$$

These results do not come as a surprise. Indeed, it is well known that, if the yield surface does not strictly contain the origin and the generalized normality rule applies, then the dissipation is zero and the dissipation pseudo-potential is the cone dual to the cone of admissible stresses K (see Figure 7). A typical example where the dissipation is also zero is given by the cohesionless Mohr–Coulomb model with an associated flow rule. Accordingly, to have a non-zero dissipation, we need a non-standard law as the one provided by the modified Cam-clay model itself. The functions $\psi(\dot{\boldsymbol{\kappa}})$ and $\psi^*(\boldsymbol{\pi})$ satisfy the following relation:

$$\psi(\dot{\boldsymbol{\kappa}}') + \psi^*(\boldsymbol{\pi}') \geq \boldsymbol{\pi}' \cdot \dot{\boldsymbol{\kappa}}', \quad \forall \left(\boldsymbol{\pi}', \dot{\boldsymbol{\kappa}}'\right) \in \mathcal{V} \times \mathcal{F}. \tag{85}$$

A pair $(\boldsymbol{\pi}, \dot{\boldsymbol{\kappa}})$ related by the generalized normality satisfies

$$\dot{\boldsymbol{\kappa}} \in \partial\psi^*(\boldsymbol{\pi}) \Leftrightarrow \boldsymbol{\pi} \in \partial\psi(\dot{\boldsymbol{\kappa}}) \Leftrightarrow \psi(\dot{\boldsymbol{\kappa}}) + \psi^*(\boldsymbol{\pi}) = \boldsymbol{\pi} \cdot \dot{\boldsymbol{\kappa}}. \tag{86}$$

Using the yield function as given by (65), which is homogeneous of degree one, we can see that the dissipation is zero by simply applying the generalized normality and using the Euler identity

$$\mathcal{D} = \boldsymbol{\pi} \cdot \dot{\boldsymbol{\kappa}} = \boldsymbol{\pi} \cdot \dot{\lambda} \frac{\partial f}{\partial \boldsymbol{\pi}} = \dot{\lambda}\, f(\boldsymbol{\pi}) = 0,$$

since $f(\boldsymbol{\pi}) = 0$. If the origin is inside the convex domain K in the generalized stresses space, it is possible to express the region K as a level set $\{\boldsymbol{\pi} : \gamma_K(\boldsymbol{\pi}) \leq 1\}$ where γ_K is a nonnegative, positively homogeneous convex function called the *gauge*. The mathematical definition of the gauge is

$$\gamma_K(\boldsymbol{\pi}) = \inf\{\mu > 0 : \boldsymbol{\pi} \in \mu K\}$$

and the dissipation is now given by

$$\mathcal{D} = \dot{\lambda},$$

where the plastic multiplier $\dot{\lambda}$ is obtained using $\gamma_K(\boldsymbol{\pi})$.

8. Implicit normality rule

The modified Cam-clay model is non-standard, but we will see below that it is still possible to obtain a variational formulation of the evolution law. In the original model, the non-normality is partial and concerns only the internal variable α. Applying the following change of variables

$$-\dot{\vartheta} = -\left(\dot{\alpha} - M\, \|\dot{\boldsymbol{\varepsilon}}\|^p\right), \tag{87}$$

we can recover the normality rule

$$\dot{\boldsymbol{\varepsilon}}^p \in \partial_{\boldsymbol{\sigma}} \psi^*(\boldsymbol{\sigma}, p_c) \quad \text{and} \quad -\dot{\vartheta} \in \partial_{p_c} \psi^*(\boldsymbol{\sigma}, p_c) \tag{88}$$

or in more compact form

$$\dot{\boldsymbol{\xi}} \in \partial\psi^*(\boldsymbol{\pi}), \tag{89}$$

where the vector $\dot{\boldsymbol{\xi}}$ is given by

$$\dot{\boldsymbol{\xi}} = \begin{bmatrix} \dot{\boldsymbol{\varepsilon}}^p \\ -\dot{\vartheta} \end{bmatrix}. \tag{90}$$

The inverse law is

$$\boldsymbol{\pi} \in \partial\psi(\dot{\boldsymbol{\xi}}), \tag{91}$$

where $\psi(\dot{\boldsymbol{\xi}})$ is the indicator function of K^* which now depends on $\dot{\boldsymbol{\varepsilon}}^p$ and $-\dot{\vartheta}$

$$K^* = \{\dot{\boldsymbol{\xi}} \in \mathcal{V} \mid M \, \|\dot{\boldsymbol{\varepsilon}}^p\|^*_{cc} + \dot{\vartheta} - \dot{e}^p_m \leq 0\}. \tag{92}$$

Here $\psi(\dot{\boldsymbol{\xi}})$ and $\psi^*(\boldsymbol{\pi})$ satisfy the Fenchel inequality

$$\psi(\dot{\boldsymbol{\xi}}') + \psi^*(\boldsymbol{\pi}') \geq \boldsymbol{\pi}' \cdot \dot{\boldsymbol{\xi}}', \quad \forall \left(\boldsymbol{\pi}', \dot{\boldsymbol{\xi}}'\right) \in \mathcal{V} \times \mathcal{F}. \tag{93}$$

Although this relation provides further insight into such a plastic model, additional developments can still be made to establish a relationship between $\dot{\boldsymbol{\kappa}}$ and $\boldsymbol{\pi}$ based on a normality rule. To recover a relation between the dual variables $\dot{\boldsymbol{\kappa}}$ and $\boldsymbol{\pi}$, we add $\boldsymbol{\pi}' \cdot \dot{\boldsymbol{\kappa}}'$ to both sides of (93),

$$\psi(\dot{\boldsymbol{\xi}}') + \psi^*(\boldsymbol{\pi}') + \boldsymbol{\pi}' \cdot (\dot{\boldsymbol{\kappa}}' - \dot{\boldsymbol{\xi}}') \geq \boldsymbol{\pi}' \cdot \dot{\boldsymbol{\kappa}}', \quad \forall (\boldsymbol{\pi}', \dot{\boldsymbol{\xi}}') \in \mathcal{V} \times \mathcal{F}. \tag{94}$$

The left-hand side of (94) is a function of both $\dot{\boldsymbol{\kappa}}'$ and $\boldsymbol{\pi}'$, which cannot be represented as the sum of two functions, one of $\dot{\boldsymbol{\kappa}}'$ and another of $\boldsymbol{\pi}'$. We call this function a *bi-potential* and its general expression is given by

$$b_p(\dot{\boldsymbol{\xi}}', \boldsymbol{\pi}') := \psi(\dot{\boldsymbol{\xi}}') + \psi^*(\boldsymbol{\pi}') + \boldsymbol{\pi}' \cdot (\dot{\boldsymbol{\kappa}}' - \dot{\boldsymbol{\xi}}'). \tag{95}$$

The right-hand side of (95) is developed using the change of variables (87). The cone K^* is not the dual of K anymore. Its expression is given by

$$K^* = \{\dot{\boldsymbol{\kappa}} \in \mathcal{V} \mid \dot{\alpha} - \dot{e}^p_m \leq 0\}. \tag{96}$$

Developing the scalar product in (95), we obtain the bi-potential for the modified Cam-clay model:

$$b_p(\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi}) = \Psi_K(\boldsymbol{\pi}) + \Psi_{K^*}(\boldsymbol{\kappa}) - M \, p_c \|\dot{\boldsymbol{\varepsilon}}^p\|^*_{cc}. \tag{97}$$

The bi-potential is positive function and satisfies the fundamental inequality

$$b_p(\dot{\boldsymbol{\kappa}}', \boldsymbol{\pi}') \geq \dot{\boldsymbol{\kappa}}' \cdot \boldsymbol{\pi}'. \tag{98}$$

A strict equality is obtained in (98) for any pair $(\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi})$ related by the evolution law:

$$b_p(\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi}) = \dot{\boldsymbol{\kappa}} \cdot \boldsymbol{\pi}. \tag{99}$$

The relations (98) and (99) can be combined to give

$$\forall \boldsymbol{\pi}' \in \mathcal{F}: \quad b_p(\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi}') - b_p(\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi}) \geq \dot{\boldsymbol{\kappa}} \cdot (\boldsymbol{\pi}' - \boldsymbol{\pi}), \tag{100}$$

$$\forall \dot{\boldsymbol{\kappa}}' \in \mathcal{V}: \quad b_p(\dot{\boldsymbol{\kappa}}', \boldsymbol{\pi}) - b_p(\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi}) \geq \boldsymbol{\pi} \cdot (\dot{\boldsymbol{\kappa}}' - \dot{\boldsymbol{\kappa}}), \tag{101}$$

which means that

- the bi-potential is bi-convex that is $b_p(\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi})$ is a convex function of $\dot{\boldsymbol{\kappa}} \in \mathcal{V}$ for each $\boldsymbol{\pi} \in \mathcal{F}$ and a convex function of $\boldsymbol{\pi} \in \mathcal{F}$ for each $\dot{\boldsymbol{\kappa}} \in \mathcal{V}$;
- the evolution law and its inverse derive from the bi-potential $b_p(\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi})$

$$\dot{\boldsymbol{\kappa}} \in \partial_{\boldsymbol{\pi}} b_p(\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi}) \quad \text{and} \quad \boldsymbol{\pi} \in \partial_{\dot{\boldsymbol{\kappa}}} b_p(\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi}). \tag{102}$$

The advantage of the present formulation results in a compact form of the evolution law formulated with one variational inequality. This formulation of the evolution law can be advantageously exploited to derive a robust algorithm. The relations (102) are essential for the derivation of stationary principles involving a functional that depends now on both the velocities and the stresses.

9. Conclusions

An important task for an engineer is to assess the stability of a geotechnical structure. Nowadays this task is carried out using finite-element codes in conjunction with complex constitutive models. For most practical cases (small displacements and small deformations), the nature of the constitutive operator has a significant influence on the convergence of numerical algorithms. It has been recognized that a variational formulation has several advantages, among them, the possibility to associate extremum (or at least stationary) principles to weak formulations of the initial/boundary-value problems. It also permits to express the behavior in a succinct manner. Indeed the stress-strain relationship derives from a scalar-valued function which acts as a potential. For plastic models such a property exists if the maximum dissipation principle holds. However, geomaterial models do not exhibit such a strong variational structure. By allowing an implicit form of the evolution rule, one may recover a weaker variational formulation and the pseudo-potential concept (introduced by Moreau) can be extended to cover non-standard behaviors. The pseudo-potential is replaced by the bi-potential, which depends on both the generalized stresses and the velocities. The bi-potential is not convex but bi-convex, which means convex with respect to the generalized stresses and the plastic strain rates when considered separately. The partial sub-derivatives of the bi-potential yield the evolution law and its inverse. It has been shown that the evolution law of the modified Cam-clay model can be derived from a bi-potential which serves as a "potential" for both the generalized stresses and the velocities. As a consequence, coupled extremum principles exist. These principles are not as strong as the usual extremum (or stationary) principles since they involve static and kinematic variables, but at least they provide new insights into this difficult problem. Further research has to be carried out to design new algorithms using, for instance, mixed formulations.

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